17. 09. 2020

Today :

- Sheaves & Čech cohomology
- Classification of (holomorphic) line bundles
- Elliptic curves & Weierstraß ℘-function.

§ Sheaves & Čech cohomology.

Let X : topological space

<u>Def</u> A sheaf $\mathcal{F}$ (of abelian groups, $\mathbb{C}$-v.sp, ...) on X is an assignment

$$\underset{\text{open}}{U \subseteq X} \longmapsto \mathcal{F}(U) \quad \nwarrow \text{abelian group, etc}$$

together with restriction maps

$$U \subseteq V, \quad r_{VU} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$
$$\sigma \longmapsto \sigma|_U$$

satisfying the following properties

(i) $U \subseteq V \subseteq W$,

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "r_{WV}"] \arrow[rr, bend right, "r_{WU}"'] & \mathcal{F}(V) \arrow[r, "r_{VU}"] & \mathcal{F}(U) \end{tikzcd}$$

(ii) $U, V \subseteq X$ $\quad \sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ s.t

$$\sigma|_{U\cap V} = \tau|_{U\cap V}$$

then there exists $\rho \in \mathcal{F}(U \cup V)$. $\rho|_U = \sigma$ & $\rho|_V = \tau$.

(iii) if $\sigma \in \mathcal{F}(U \cup V)$ with $\sigma|_U = \sigma|_V = 0$, then

$$\sigma = 0.$$

Eg1 X : complex manifold. The constant sheaf $\mathbb{Z}$ is the sheafification of the following presheaf

$$U \longmapsto \mathbb{Z} \quad \text{when } U \neq \phi \subseteq X.$$

$$\phi \longmapsto 0$$

Eg2 $L \xrightarrow{\pi} X$ holomorphic line on a complex analytic manifold. Then we can associate a sheaf $\mathcal{L}$ by

$$U \longmapsto \{\text{holomorphic sections on } L|_U\}.$$

By the abuse of notation, we interchangeably write

$$\mathcal{L} \longleftrightarrow L.$$

Eg3 X : algebraic variety w/ Zariski topology.
Then by definition, we have the structure sheaf $\mathcal{O}_X$. (or $\mathcal{O}_X^{alg}$)

Čech cohomology.

Let $\underline{\mathcal{U}} = \{U_\alpha\}_{\alpha \in \Lambda}$ be a locally finite open cover of X, $\mathcal{F}$: sheaf on X

We define the Čech complex of $\mathcal{F}$ associated to $\underline{\mathcal{U}}$ as follows :

$$C^0(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha \in \Lambda} \mathcal{F}(U_\alpha)$$

$$C^1(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha \neq \beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

$$\vdots$$

$$C^p(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha_{i_0} \neq \cdots \neq \alpha_{i_p}} \mathcal{F}(U_{\alpha_{i_0}} \cap \cdots \cap U_{\alpha_{i_p}})$$

and

$$\delta : C^p(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})$$

$$(\delta\sigma)_{\alpha_{i_0} \cdots \alpha_{i_{p+1}}} = \sum_{j=0}^{p+1} (-1)^j \, \sigma_{\alpha_{i_0} \cdots \widehat{\alpha_{i_j}} \cdots \alpha_{i_{p+1}}} \Big|_{U_{\alpha_{i_0}} \cap \cdots \cap U_{\alpha_{i_{p+1}}}}$$

It is an easy exercise to check that

$$\cdots \xrightarrow{\delta} C^{p+1}(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\delta} C^p(\underline{\mathcal{U}}, \mathcal{F}) \xrightarrow{\delta} C^{p+1}(\underline{\mathcal{U}}, \mathcal{F}) \to \cdots$$

is a chain complex, i.e. $\delta^2 = 0$.

<u>Def</u> $$H^p(\underline{\mathcal{U}}, \mathcal{F}) = \frac{\ker\left(C^p(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow C^{p+1}(\underline{\mathcal{U}}, \mathcal{F}) \right)}{\mathrm{Im}\left(C^{p-1}(\underline{\mathcal{U}}, \mathcal{F}) \to C^p(\underline{\mathcal{U}}, \mathcal{F}) \right)}$$

If $\underline{\mathcal{V}}$ is a refinement of $\underline{\mathcal{U}}$, then there exists

$$H^p(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow H^p(\underline{\mathcal{V}}, \mathcal{F})$$

via restriction maps

<u>Def</u> (Čech cohomology)

$$H^q(X, \mathcal{F}) = \varinjlim_{\underline{\mathcal{U}}} H^q(\underline{\mathcal{U}}, \mathcal{F})$$

<u>Ex</u> $H^0(X, \mathcal{F}) = \mathcal{F}(X)$. (def of sheaves)

When we compute Čech cohomology, it is enough to take an acyclic cover of $\mathcal{F}$.

<u>Def</u> $\underline{\mathcal{U}}$ is called an acyclic cover of $\mathcal{F}$ if

$$H^q(U_{\alpha_{i_0}} \cap \cdots \cap U_{\alpha_{i_p}}, \mathcal{F}) = 0$$

for all $q > 0$, $\alpha_{i_0}, \cdots \alpha_{i_p}$

<u>Thm</u> (Leray) If $\underline{\mathcal{U}}$ is an acyclic cover of $\mathcal{F}$,

$$H^q(\underline{\mathcal{U}}, \mathcal{F}) = H^q(X, \mathcal{F}).$$

Ex1. Let $D \overset{\text{open}}{\subseteq} \mathbb{C}$ be a connected, simply connected domain. Then

$$H^q(D, \mathcal{O}_D^{an,*}) = 0 \quad \forall q > 0.$$

Ex2 If X is a paracompact space, then

$$\underset{\text{Čech cohomology}}{\check{H}^*(X, \underline{\mathbb{Z}})} \cong \underset{\text{singular cohomology.}}{H^*(X, \mathbb{Z})}$$

Pf) [Griffith-Harris, p42] ← when X is compact

There is another definition of a sheaf cohomology coming from an injective resolution of $\mathcal{F}$.

↳ See [Hartshorne, Ch3]

§ Classification of (holomorphic) line bundles

Let $(X, \mathcal{O}_X^{an})$ be a complex manifold with the sheaf of holomorphic functions.

<u>Goal</u> There exists a bijective map

$$\{\text{line bundles on } X\}/\sim \xrightarrow{1:1} H^1(X, \mathcal{O}_X^*)$$

$\mathcal{O}_X^*$: sheaf of invertible functions.

How to construct a map?

$L \to X$ holomorphic line bundle

$\rightsquigarrow$ $\exists$ open cover $\{U_\alpha\}$ of X st $L|U_\alpha$: trivial

$$\varphi_\alpha : L|U_\alpha \xrightarrow{\cong} U_\alpha \times \mathbb{C}$$

$$\begin{tikzcd}
 & L|_{U_\alpha \cap U_\beta} \arrow[dl, "\varphi_\beta"', "\cong"] \arrow[dr, "\varphi_\alpha", "\cong"'] & \\
(U_\alpha \cap U_\beta) \times \mathbb{C} \arrow[rr, "Id \times g_{\alpha\beta}"] & & (U_\alpha \cap U_\beta) \times \mathbb{C}
\end{tikzcd}$$

where $\quad g_{\alpha\beta} = (\varphi_\alpha \circ \varphi_\beta^{-1})|_{U_\alpha \cap U_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

$g_{\alpha\beta}$ satisfies the cocycle conditions:

$$\begin{cases} g_{\alpha\beta} \circ g_{\beta\alpha} = Id \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \circ g_{\gamma\alpha} = Id \end{cases} \qquad (*)$$

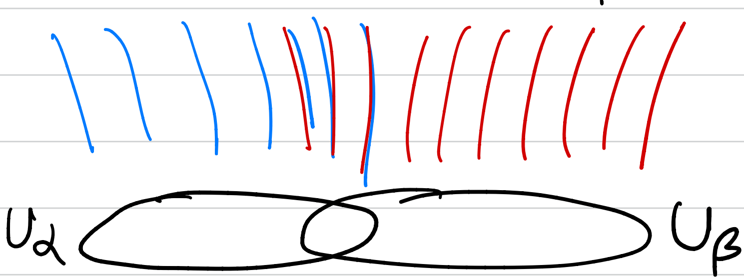
<u>Check</u> The condition $(*)$ implies that $[g_{\alpha\beta}] \in C^1(\underline{\mathcal{U}}, \mathcal{O}^*)$ satisfies

$$\delta[g_{\alpha\beta}] = 0 \quad \text{in } C^2(\underline{\mathcal{U}}, \mathcal{O}_X^*)$$

$\Rightarrow [g_{\alpha\beta}] \in H^1(X, \mathcal{O}_X^*)$

On the other hand, if $\{g_{\alpha\beta}\}$ satisfies $(*)$ one can construct a line bundle by

$$L = \coprod_{\alpha \in \Lambda} U_\alpha \times \mathbb{C} \Big/ \sim g_{\alpha\beta}$$



$$L \xrightarrow[\cong]{f} L'$$

Two different translation functions

$$\{g_{\alpha\beta}\}_{\alpha,\beta\in\Lambda} \quad \& \quad \{g'_{\alpha\beta}\}_{\alpha,\beta\in\Lambda}$$

on a covering $\{U_\alpha\}_{\alpha\in\Lambda}$ define isomorphic line bundles iff

$$\exists\ f_\alpha \in \mathcal{O}^*(U_\alpha) \quad \text{s.t} \qquad g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta}\, g_{\alpha\beta}$$

In the language of Čech cohomology,

$$[g_{\alpha\beta}] - [g'_{\alpha\beta}] = \delta[f_\alpha] \qquad C^1(\underline{\mathcal{U}}, \mathcal{O}_X^*)$$

Thus we find a map

$$\{\text{line bundles on } X\}/_\sim \longrightarrow H^1(X, \mathcal{O}_X^*)$$

$$L_1 \otimes L_2 \qquad [\mathcal{L}] \longmapsto [g_{\alpha\beta}]$$

??

with the inverse $L^\vee$

$$[g_{\alpha\beta}] \longmapsto L = \coprod U_\alpha \times \mathbb{C} /\sim$$

There is a subtle point to settle the comparison

$$\begin{tikzcd} \text{Algebraic Curve} \arrow[r, bend left, "\text{trivial}"] & \text{Riemann surface} \arrow[l, bend left, "\text{analysis}"] \end{tikzcd}$$

which can be overlooked. This comparison is only true for projective / complete / compact curves. For instance for non compact situation algebraic curves & Riemann surfaces behave differently.

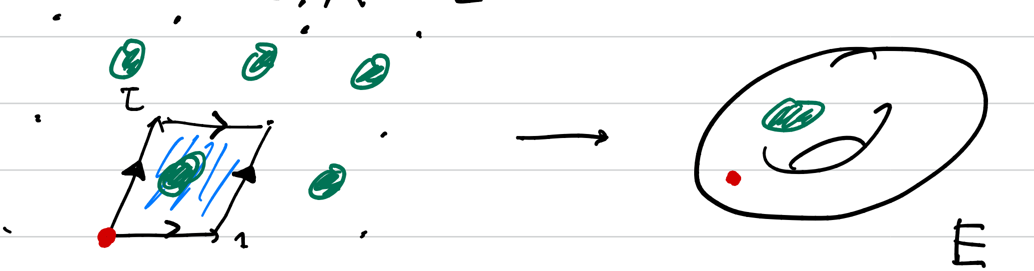
Q) Can you justify this statement?

§ Elliptic curves & Weierstraß functions.

Recall : We defined elliptic curves as complex 1-dim'l manifold.

$\Lambda = \mathbb{Z}1 \oplus \mathbb{Z}\tau \subset \mathbb{C}$: rank 2 lattice. $\tau \in \mathbb{H}$ ← upper half plane

$$\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Lambda =: E$$



Goal : Understand E as a projective variety.
In fact, there exists a degree 3 homogeneous polynomial

$$F \in \mathbb{C}[X_0, X_1, X_2]$$

and a map

$$\varphi : E \longrightarrow \mathbb{P}^2_{\mathbb{C}}$$

where the image is a smooth cubic $V(F)$.

<u>Def</u> (Weierstraß)

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \quad z \notin \Lambda$$

$$\Lambda^* = \Lambda - \{0\}$$

<u>Ex</u> $\wp(z)$ is a meromorphic function with double poles at Λ

(so one has to show convergence of $\wp$. Moreover, one can show $\wp$ converges uniformly on each compact subset of $\mathbb{C} \setminus \Lambda$).

Now

$$\wp'(z) = -\sum_{\omega \in \Lambda} \frac{-2}{(z-\omega)^3}$$

which is

$$\begin{cases} \wp'(z) = \wp'(z+\omega_0) & \omega_0 \in \Lambda \\ \wp'(-z) = -\wp'(z) \end{cases}.$$

It is not hard to show

$$\begin{cases} \wp(z) = \wp(z+\omega_0) & \omega_0 \in \Lambda \\ \wp(-z) = \wp(z) \end{cases}$$

<u>Claim</u> $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ where

$$g_2 = 60\sum_{\omega\in\Lambda^*}\frac{1}{\omega^4}, \quad g_3 = 140\sum_{\omega\in\Lambda^*}\frac{1}{\omega^6}$$

Sketch) Consider the Laurent expansion of $\wp$ around $z=0$:

$$\wp(z) = \frac{1}{z^2} + 0 + \frac{1}{20}g_2 z^2 + \frac{1}{28}g_3 z^3 + O(z^6)$$

(bc $\wp$ is even) and similarly for $\wp'(z)$.

Simple computation shows that

$$(\wp')^2 - 4\wp^3 + g_2\wp + g_3$$

is holomorphic on $\mathbb{C}$ & Λ-periodic. Since holomorphic function on a compact domain is constant, it is constant (in fact 0)

Δ

Let

$$E = \mathbb{C}/\Lambda \xrightarrow{\varphi} \mathbb{P}^2_{\mathbb{C}} \quad z \longmapsto [\wp(z) : \wp'(z) : 1], \; z \neq 0$$
$$0 \longmapsto [0:1:0].$$

Then φ factors through $V(F)$ where

$$F(X_0, X_1, X_2) = X_1^2 X_2 - 4X_0^3 + g X_0 X_2^2 + g_3 X_3^3$$

Now it is not hard to prove that $V(F)$ is a nonsingular & connected curve.

compute partial differentials to show that F is non singular $\Longleftrightarrow g_2^3 - 27 g_3^2 \neq 0$. Now $g_2^3 - 27 g_3^2$ is the discriminant of F. Why it does not vanish?

<u>Check</u> (i) φ is surjective onto $V(F)$

↳ basically open mapping thm

(ii) φ is injective.

(the following observation can be useful: let $z_1, z_2 \in \mathbb{C}$. Then

$$\wp(z_1) = \wp(z_2) \Longleftrightarrow z_1 + z_2 \in \Lambda \text{ or } z_1 - z_2 \in \Lambda.$$
)

ASIDE: For higher genus Riemann surfaces:
$(g \geq 2)$

$$\mathbb{H} \longrightarrow X$$

$X \cong \mathbb{H}/\Gamma$ $\quad \Gamma \leq PSL(2,\mathbb{Z})$ discrete subgroup with some properties (Fuchsian group).

To understand the "algebraization" we need Chow's Theorem (see [Griffith-Harris]

Harder part